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Nonlinear harmonic oscillators

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Abstract

The existence is noted of assemblies of an arbitrary number of complex oscillators, or equivalently, of an arbitrary even number of real oscillators, characterized by Newtonian equations of motion ('acceleration equal force') with one-body velocity-dependent linear forces and many-body velocity-independent cubic forces, *all* the nonsingular solutions of which are *isochronous* (completely periodic with the same period). As for the singular solutions, as usual they emerge, in the context of the initial-value problem, from a closed domain in phase space having lower dimensionality.

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1. Introduction

This paper advertises the existence of assemblies of an arbitrary number of complex oscillators, or equivalently of an arbitrary even number of real oscillators, characterized by Newtonian equations of motion ('acceleration equal force') with one-body velocity-dependent linear forces and many-body velocity-independent cubic forces, *all* the nonsingular solutions of which are *isochronous* (completely periodic with the same period). As for the singular solutions, as usual they emerge, in the context of the initial-value problem, from a closed domain in phase space having lower dimensionality.

It seems appropriate to denote such *nonlinear* oscillators as *harmonic*, since the original meaning of this adjective denotes the absence of the cacophony associated with nonperiodic (including multiply periodic) phenomena. The association of the attributes 'nonlinear' and 'harmonic'—as made in the title of this paper—might however sound oxymoronic due to the widespread idea that nonlinear many-degrees-of-freedom systems generally behave in a nonisochronous manner (i.e., not in a completely periodical fashion, or, even if periodically, then with periods that in the context of the initial-value problem do depend on the initial data); this conviction indeed causes the two words 'linear' and 'harmonic' to be often used as synonyms. Yet, once their origin is understood, the results reported in this paper—which clearly question the belief outlined above—shall become quite obvious, even trivial—as is eventually the case for all mathematically correct results.

The results reported below are a rather immediate consequence of two recent developments: a 'trick' suitable to manufacture/uncover systems possessing many periodic solutions [1–3], recently utilized in several contexts [4–18]; and the identification of several many-body systems of Newtonian type amenable to exact treatments via their relation with matrix evolution equations [19–21, 4]. Indeed, the results reported herein are analogous to, albeit remarkably more cogent than, some previous findings (see for instance the last two exercises of [4]; but beware of a misprint there, in equation (5.6.5-41a) $2/p$ should read $p/2$); the purpose and scope of the present paper is to formulate them in a precise manner and especially to back them with mathematically rigorous proofs (also based on previous results [22–24]).

Various avatars of these results are stated in section 2 and then proved in section 3. Final remarks are proffered in section 4.

2. Results

Notation: Throughout this paper, the independent variable t is real ('time'), and ω is a real (without loss of generality, positive, $\omega > 0$) constant ('circular frequency') to which the 'period'

$$T = \frac{2\pi}{\omega} \tag{2.1}$$

is associated.

Lemma 2.1. All *nonsingular solutions of the following complex matrix evolution equation,*

$$\ddot{M} - 3\mathrm{i}\omega\dot{M} - 2\omega^2 M = cM^3 \tag{2.2}$$

are completely periodic *with period T (isochronous!),*

$$M(t+T) = M(t). \tag{2.3}$$

Here (and always below) $M \equiv M(t)$ is a complex square matrix of *arbitrary* rank and c is an arbitrary (possibly complex) scalar constant.

The singular solutions of (2.2) are not generic, namely—in the context of the initial-value problem—they obtain only for special values of the initial data $M(0)$, $\dot{M}(0)$ characterized by the requirement to satisfy certain equalities (equalities, not inequalities—as entailed by the proof of this *lemma 2.1*, see section 3).

Corollary 2.2. *Likewise,* all *nonsingular solutions of the following systems of two coupled real matrix evolution equations,*

$$\ddot{U} + 3\omega\dot{V} - 2\omega^2 U = a(U^3 - UV^2 - V^2U - VUV) - b(U^2V + V^2U - V^3 + UVU) \tag{2.4a}$$

$$\ddot{V} - 3\omega\dot{U} - 2\omega^2 V = b(U^3 - UV^2 - V^2U - VUV) + a(U^2V + V^2U - V^3 + UVU) \tag{2.4b}$$

are completely periodic *with period T (isochronous!),*

$$U(t+T) = U(t) \qquad V(t+T) = V(t). \tag{2.5}$$

Here $U \equiv U(t)$, $V \equiv V(t)$ are two real square matrices of arbitrary rank (the same for both of them) and a, b are two arbitrary real (scalar) constants.

Clearly this *corollary 2.2* corresponds to the preceding *lemma 2.1* via the identification of the two matrices U, V as the real and imaginary parts of the matrix M, $M = U + \mathrm{i}V$ and likewise of the two scalar constants a, b as the real and imaginary parts of the constant c, $c = a + \mathrm{i}b$.

Remark 2.3. A trivial generalization of the result of *lemma 2.1* is obtained by replacing the complex matrix evolution equation (2.2) with, say,

$$\ddot{M} - 3\mathrm{i}\omega\dot{M} - 2\omega^2 M = MCMCM \tag{2.6}$$

where $M \equiv M(t)$ is now a rectangular $(N \times L)$-matrix while C is a (complex) constant rectangular $(L \times N)$-matrix, with N, L two arbitrary positive integers; indeed (2.6) is obtained from (2.2) via the replacement of M with CM. An additional trivial generalization is obtained by adding a constant matrix to M. Analogous generalizations can obviously be made of *corollary 2.2*.

Proposition 2.4. All *the nonsingular solutions of the following complex Newtonian equations of motion*

$$\ddot{\vec{r}}_k^{(l)} - 3\mathrm{i}\omega\dot{\vec{r}}_k^{(l)} - 2\omega^2\vec{r}_k^{(l)} = c\sum_{l'=1}^{L}\sum_{k'=1}^{K}\vec{r}_k^{(l')}\left(\vec{r}_{k'}^{(l')}\cdot\vec{r}_{k'}^{(l)}\right) \tag{2.7}$$

or equivalently (via $\vec{r}_k^{(l)} = \vec{u}_k^{(l)} + \mathrm{i}\vec{v}_k^{(l)}$, $c = a + \mathrm{i}b$*) of the following real Newtonian equations of motion:*

$$\begin{aligned}\ddot{\vec{u}}_k^{(l)} + 3\omega\dot{\vec{v}}_k^{(l)} - 2\omega^2\vec{u}_k^{(l)} = {}& a\sum_{l'=1}^{L}\sum_{k'=1}^{K}\left\{\vec{u}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right) - \left(\vec{v}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right)\right] - \vec{v}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right)\right.\right.\\ &\left.\left.+\left(\vec{v}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right)\right]\right\} - b\sum_{l'=1}^{L}\sum_{k'=1}^{K}\left\{\vec{u}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right) + \left(\vec{v}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right)\right]\right.\\ &\left.+\vec{v}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right) - \left(\vec{v}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right)\right]\right\}\end{aligned} \tag{2.8a}$$

$$\begin{aligned}\ddot{\vec{v}}_k^{(l)} - 3\omega\dot{\vec{u}}_k^{(l)} - 2\omega^2\vec{v}_k^{(l)} = {}& b\sum_{l'=1}^{L}\sum_{k'=1}^{K}\left\{\vec{u}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right) - \left(\vec{v}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right)\right] - \vec{v}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right)\right.\right.\\ &\left.\left.+\left(\vec{v}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right)\right]\right\} + a\sum_{l'=1}^{L}\sum_{k'=1}^{K}\left\{\vec{u}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right) + \left(\vec{v}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right)\right]\right.\\ &\left.+\vec{v}_k^{(l')}\left[\left(\vec{u}_{k'}^{(l')}\cdot\vec{u}_{k'}^{(l)}\right) - \left(\vec{v}_{k'}^{(l')}\cdot\vec{v}_{k'}^{(l)}\right)\right]\right\}\end{aligned} \tag{2.8b}$$

are completely periodic *with period T (isochronous!):*

$$\vec{r}_k^{(l)}(t+T) = \vec{r}_k^{(l)}(t) \tag{2.9a}$$

$$\vec{u}_k^{(l)}(t+T) = \vec{u}_k^{(l)}(t) \qquad \vec{v}_k^{(l)}(t+T) = \vec{v}_k^{(l)}(t). \tag{2.9b}$$

Here $l = 1, \ldots, L$ and $k = 1, \ldots, K$, with L and K two arbitrary positive integers, $\vec{r}_k^{(l)} \equiv \vec{r}_k^{(l)}(t)$, respectively, $\vec{u}_k^{(l)} \equiv \vec{u}_k^{(l)}(t)$, $\vec{v}_k^{(l)} \equiv \vec{v}_k^{(l)}(t)$ denote complex respectively real

S-vectors, with S an arbitrary positive integer and the dots sandwiched among two vectors denote the standard Euclidean scalar product in S-dimensional space.

Proposition 2.5. All *the nonsingular solutions of the following complex Newtonian equations of motion:*

$$\ddot{\vec{r}}_k^{(l)} - 3\mathrm{i}\omega\dot{\vec{r}}_k^{(l)} - 2\omega^2\vec{r}_k^{(l)} = c\sum_{l'=1}^{L}\sum_{k'=1}^{K}\vec{r}_{k'}^{(l')}\left(\vec{r}_{k'}^{(l')}\cdot\vec{r}_k^{(l)}\right) \tag{2.10}$$

or equivalently (via $\vec{r}_k^{(l)} = \vec{u}_k^{(l)} + \mathrm{i}\vec{v}_k^{(l)}$, $c = a + \mathrm{i}b$*) of the following real Newtonian equations of motion*

$$\begin{aligned}\ddot{\vec{u}}_k^{(l)} + 3\omega\dot{\vec{v}}_k^{(l)} - 2\omega^2\vec{u}_k^{(l)} = a\sum_{l'=1}^{L}\sum_{k'=1}^{K}\Big\{&\vec{u}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big) - \Big(\vec{v}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big)\Big] - \vec{v}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big)\\ &+\Big(\vec{v}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big)\Big]\Big\} - b\sum_{l'=1}^{L}\sum_{k'=1}^{K}\Big\{\vec{u}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big) + \Big(\vec{v}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big)\Big]\\ &+\vec{v}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big) - \Big(\vec{v}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big)\Big]\Big\}\end{aligned} \tag{2.11a}$$

$$\begin{aligned}\ddot{\vec{v}}_k^{(l)} - 3\omega\dot{\vec{u}}_k^{(l)} - 2\omega^2\vec{v}_k^{(l)} = b\sum_{l'=1}^{L}\sum_{k'=1}^{K}\Big\{&\vec{u}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big) - \Big(\vec{v}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big)\Big] - \vec{v}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big)\\ &+\Big(\vec{v}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big)\Big]\Big\} + a\sum_{l'=1}^{L}\sum_{k'=1}^{K}\Big\{\vec{u}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big) + \Big(\vec{v}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big)\Big]\\ &+\vec{v}_{k'}^{(l')}\Big[\Big(\vec{u}_{k'}^{(l')}\cdot\vec{u}_k^{(l)}\Big) - \Big(\vec{v}_{k'}^{(l')}\cdot\vec{v}_k^{(l)}\Big)\Big]\Big\}\end{aligned} \tag{2.11b}$$

are completely periodic *with period T (isochronous!), see (2.8), (2.9).*

Here we use of course the same notation as in the previous *proposition 2.4*.

Remark 2.6. These results, *proposition 2.4* and *2.5*, can obviously be generalized in analogy to the generalization of *lemma 2.1* and of *corollary 2.2* entailed by *remark 2.3*. Let us, moreover, emphasize the covariant character of these equations of motion, (2.7)–(2.11), entailing their rotation-invariant character (in S-dimensional space).

Proposition 2.7. All *the nonsingular solutions of the following complex Newtonian equations of motion:*

$$\begin{aligned}\ddot{\rho}_{nm} - 3\mathrm{i}\omega\dot{\rho}_{nm} - 2\omega^2\rho_{nm} = c\sum_{j,k=1}^{N}\{&\rho_{nj}\rho_{jk}\rho_{km} + [(\vec{r}_{nj}\wedge\vec{r}_{jk})\cdot\vec{r}_{km}]\\ &-[\rho_{nj}(\vec{r}_{jk}\cdot\vec{r}_{km}) + \rho_{jk}(\vec{r}_{nj}\cdot\vec{r}_{km}) + \rho_{km}(\vec{r}_{nj}\cdot\vec{r}_{jk})]\}\end{aligned} \tag{2.12a}$$

$$\begin{aligned}\ddot{\vec{r}}_{nm} - 3\mathrm{i}\omega\dot{\vec{r}}_{nm} - 2\omega^2\vec{r}_{nm} = +b\sum_{j,k=1}^{N}\{&\vec{r}_{nj}[\rho_{jk}\rho_{km} - (\vec{r}_{jk}\cdot\vec{r}_{km})] + \vec{r}_{jk}[\rho_{nj}\rho_{km} + (\vec{r}_{nj}\cdot\vec{r}_{km})]\\ &+\vec{r}_{km}[\rho_{jk}\rho_{jk} - (\vec{r}_{nj}\cdot\vec{r}_{jk})] - [\rho_{nj}(\vec{r}_{jk}\wedge\vec{r}_{km})\\ &+\rho_{jk}(\vec{r}_{nj}\wedge\vec{r}_{km}) + \rho_{km}(\vec{r}_{nj}\wedge\vec{r}_{jk})]\}\end{aligned} \tag{2.12b}$$

are completely periodic *with period T (isochronous!).*

Here the indices n, m run from 1 to N, the quantities $\rho_{nm} \equiv \rho_{nm}(t)$ are N^2 complex scalars, the quantities $\vec{r}_{nm} \equiv \vec{r}_{nm}(t)$ are N^2 complex three-vectors and the dots, respectively, the wedge symbols sandwiched among two three-vectors, denote the standard scalar, respectively vector products in three-dimensional space. Note that these equations of motion are covariant, hence rotation-invariant in three-dimensional space; they are moreover invariant under the parity transformation if one assumes all the dependent variables ρ_{nm} to behave under such transformation as pseudoscalars, and all the dependent variables $\vec{r}_{nm}$ as vectors (or viceversa: ρ_{nm} scalars, $\vec{r}_{nm}$ pseudovectors or axial vectors).

It is left as an easy task for the diligent reader to write the analogous real equations of motion satisfied by the $2N^2$ real scalars and the $2N^2$ real three-vectors that constitute the real and imaginary parts of ρ_{nm} and $\vec{r}_{nm}$, as well as the generalizations of these results entailed by trivial changes of dependent variables analogous to those associated with *remarks 2.3* and *2.6*.

Proposition 2.8. All *the nonsingular solutions of the following complex Newtonian equations of motion:*

$$\begin{aligned}
&\ddot{\rho}^{(nm)(j)} - 3\mathrm{i}\omega\dot{\rho}^{(nm)(j)} - 2\omega^2\rho^{(nm)(j)} \\
&\quad = c\sum_{l,p=1}^{N}\sum_{h,k=1}^{4}\sum_{q,v=1}^{4}\Big[s_{hk}^{(j)}\Big(s_{qv}^{(h)}\rho^{(nl)(q)}\rho^{(lp)(v)} + \hat{s}_{qv}^{(h)}\vec{r}^{(nl)(q)}\cdot\vec{r}^{(lp)(v)}\Big)\rho^{(pm)(k)} \\
&\qquad + \hat{s}_{hk}^{(j)}\Big(u_{qv}^{(h)}\rho^{(nl)(q)}\vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(h)}\vec{r}^{(nl)(q)}\rho^{(lp)(v)} \\
&\qquad + w_{qv}^{(h)}\vec{r}^{(nl)(q)}\wedge\vec{r}^{(lp)(v)}\Big)\cdot\vec{r}^{(pm)(k)}\Big]
\end{aligned}\qquad (2.13a)$$

$$\begin{aligned}
&\ddot{\vec{r}}^{(nm)(j)} - 3\mathrm{i}\omega\dot{\vec{r}}^{(nm)(j)} - 2\omega^2\vec{r}^{(nm)(j)} \\
&\quad = c\sum_{l,p=1}^{N}\sum_{h,k=1}^{4}\sum_{q,v=1}^{4}\Big[u_{hk}^{(j)}\Big(s_{qv}^{(h)}\rho^{(nl)(q)}\rho^{(lp)(v)} + \hat{s}_{qv}^{(h)}\vec{r}^{(nl)(q)}\cdot\vec{r}^{(lp)(v)}\Big)\vec{r}^{(pm)(k)} \\
&\qquad + \hat{u}_{hk}^{(j)}\Big(u_{qv}^{(h)}\rho^{(nl)(q)}\vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(h)}\vec{r}^{(nl)(q)}\rho^{(lp)(v)} + w_{qv}^{(h)}\vec{r}^{(nl)(q)}\wedge\vec{r}^{(lp)(v)}\Big)\rho^{(pm)(k)} \\
&\qquad + w_{hk}^{(j)}\Big(u_{qv}^{(h)}\rho^{(nl)(q)}\vec{r}^{(lp)(v)} + \hat{u}_{qv}^{(h)}\vec{r}^{(nl)(q)}\rho^{(lp)(v)} \\
&\qquad + w_{qv}^{(h)}\vec{r}^{(nl)(q)}\wedge\vec{r}^{(lp)(v)}\Big)\wedge\vec{r}^{(pm)(k)}\Big]
\end{aligned}\qquad (2.13b)$$

are completely periodic *with period T (isochronous!).*

Here the index j runs from 1 to 4, and the indices n, m run from 1 to N; the numbers $s_{nm}^{(j)}, \hat{s}_{nm}^{(j)}, u_{nm}^{(j)}, \hat{u}_{nm}^{(j)}, w_{nm}^{(j)}$ take one of the three values $0, \pm 1$ and we refer for their definition to equation (2.12) of [21]. Note that these complex Newtonian equations of motion involve the $4N^2$ (complex) scalars $\rho^{(nm)(j)} \equiv \rho^{(nm)(j)}(t)$ and the $4N^2$ (complex) three-vectors $\vec{r}^{(nm)(j)} \equiv \vec{r}^{(nm)(j)}(t)$; they are clearly covariant, hence they describe a *rotation-invariant* dynamics in three-dimensional space. It is also easily seen from the values of the numbers $s_{nm}^{(j)}, \hat{s}_{nm}^{(j)}, u_{nm}^{(j)}, \hat{u}_{nm}^{(j)}, w_{nm}^{(j)}$ (many of which vanish [21]) that the equations of motion preserve parity, provided the quantities $\rho^{(nm)(j)}$ are interpreted as pseudoscalars for $j = 1, 3$ and as scalars for $j = 2, 4$ and likewise the quantities $\vec{r}^{(nm)(j)}$ are interpreted as vectors for $j = 1, 3$ and as pseudovectors (or axial vectors) for $j = 2, 4$.

Remark 2.9. It is of course easy to generalize these equations of motion, (2.12) and (2.13), along the lines suggested by *remark 2.3* (namely by performing linear combinations with constant coefficients of, and constant additions to, the dependent variables), as well as to obtain (possibly after such generalization) from these complex equations of motion, real equations of motion by considering separately the real and imaginary parts of all the complex numbers involved.

The examples of isochronous nonlinear oscillators exhibited above (which certainly are far from exhausting all the instances in which this phenomenology manifests itself: see section 4) should suffice to validate the title of this paper, in spite of its paradoxical aspect on which we have already commented in the introductory section 1. Let us emphasize that, for all these examples, the sets of initial data that yield singular solutions always have a lower dimensionality than the sets of (generic!) data that yield isochronous motions; this is clear from the proofs of these results, which are provided (except for those that are self-evident) in section 3.

3. Proofs

In this section we prove the results reported in the preceding section 2—except of course for those which are self-evident.

The proof of the main result, namely *lemma 2.1*, is an easy consequence of the following:

Lemma 3.1. All *solutions of the following matrix evolution equation:*

$$Y'' = cY^3 \tag{3.1}$$

where we denote with τ the (complex) independent variable, the dependent variable $Y \equiv Y(\tau)$ is a $(N \times N)$-matrix (of arbitrary rank N) and of course primes denote differentiations with respect to the independent variable τ, are meromorphic *functions of the variable τ for all (finite) values of this complex variable.*

The proof of this *lemma 3.1* is an immediate consequence of the following explicit expression of the general solution of (3.1) [22]:

$$Y(\tau) = \Omega^{-1}\tilde{Y}(\tau)\Omega \tag{3.2a}$$

$$\tilde{Y}_{jk}(\tau) = (\Omega Y(0)\Omega^{-1})_{jk} \exp[(\xi_j - \tilde{\xi}_k)\tau] \frac{\Theta(A_k - A_j + V + U\tau)\Theta(V)}{\Theta(A_k - A_j + V)\Theta(V + U\tau)}. \tag{3.2b}$$

Here Ω is the $(N \times N)$-matrix that diagonalizes the matrix $T = [Y(0), Y'(0)]$, $\Omega T \Omega^{-1} = diag(t_1, \ldots, t_N)$; the $2N$ scalars $\xi_j, \tilde{\xi}_j$ and the $N + 2$ $(N^2 - N + 1)$-vectors A_j, U, V are constant (τ-independent) and are fixed by the initial data (for details, see [22]; but beware of a slight change in notation, corresponding to a rescaling of the dependent variable Y by a factor $(-c/2)^{1/2}$), and $\Theta(W)$ is the $(N^2 - N + 1)$-dimensional theta function associated with the Riemann surface Γ defined by the spectral curve $S(z, h) = \det(L(h) - Iz) = 0$, where I is the unit $(2N \times 2N)$-matrix and $L(h)$ is the Lax $(2N \times 2N)$-matrix such that (3.1) is equivalent to the Lax equation $L' = [L, A]$. These two matrices therefore have the explicit block-matrix expressions [22]

$$L = \begin{pmatrix} Y' & Y^2 + \mathrm{i}2^{1/2}hY - h^2 I \\ Y^2 - \mathrm{i}2^{1/2}hY - h^2 I & -Y' \end{pmatrix} \tag{3.2c}$$

$$A = \begin{pmatrix} 0 & Y + \mathrm{i}2^{-1/2}hI \\ -Y + \mathrm{i}2^{-1/2}hI & 0 \end{pmatrix}. \tag{3.2d}$$

Note that while this expression (3.2) of the general solution of (3.1) is rather complicated, it clearly implies that the $(N \times N)$-matrix $Y(\tau)$ is a meromorphic function (with only simple poles), because theta functions are entire and only have simple zeros.

To prove *lemma 2.1* one now applies the 'trick' [1–18], namely the following simple change of dependent and independent variables:

$$M(t) = \exp(\mathrm{i}\omega t)Y(\tau) \qquad \tau = \frac{[\exp(\mathrm{i}\omega t) - 1]}{(\mathrm{i}\omega)}. \tag{3.3}$$

It is then easy to verify that (3.1) entails (2.2), while, of course, the fact that τ is a periodic function of t (see the second relation (3.3)) together with *lemma 3.1* clearly entails the validity of *lemma 2.1*, which is thereby proven.

Propositions 2.4, *2.5* and *2.7* are then an immediate consequence of the fact that the Newtonian equations of motion (2.7), (2.10) and (2.12) are reductions of the matrix evolution equation (2.2), corresponding to appropriate parametrizations of the matrix M as shown in [20] and reported in [4] (see in particular, section 5.6.5 of this book). Likewise, *proposition 2.8* is an immediate consequence of the fact that the equations of motion (2.13) are reductions of the matrix evolution equation (2.2) corresponding to appropriate parametrizations of the matrix M, as shown in [21].

4. Outlook

The treatment of section 3 entails the possibility to write in explicit form—in the guise of ratios of multidimensional theta functions, see (3.2) and (3.3)—the solutions of all the Newtonian equations of motion exhibited in this paper. This opens the way for more detailed analyses of the actual behaviour of the nonlinear oscillators described by these equations of motion—including the special reductions of these equations which we have not discussed in this paper but are immediately entailed by the results of [20] and [4]. We may pursue this line of research in future publications—perhaps in applicative contexts. In this connection, the existence of a much simpler subset of solutions of (2.2) should also be noted, namely those corresponding to the 'separatrix' solution of (3.1):

$$M(t) = \exp(\mathrm{i}\omega t)\left\{[M(0)]^{-1} + \left(\frac{c}{2}\right)^{1/2}\frac{[1 - \exp(\mathrm{i}\omega t)]}{(\mathrm{i}\omega)}\right\}^{-1}. \tag{4.1}$$

This solution of (2.2) is obviously completely periodic with period T, see (2.1), unless it is singular, and it is singular for real t if and only if the initial data are such that there hold (one of the) conditions $|1 + \mathrm{i}\omega\mu^{-1}(2/c)^{1/2}| = 1$, where μ denotes (any) one of the eigenvalues of the matrix $M(0)$.

An analogous treatment to that provided in this paper for nonlinear oscillators characterized by (linear and) cubic forces can be made for oscillators characterized by (linear and) quadratic forces. Indeed, there holds in this case an analogous result to *lemma 2.1*, in the guise of the following:

Lemma 4.1. All *nonsingular solutions of the following complex matrix evolution equation:*

$$\ddot{M} - 5\mathrm{i}\omega\dot{M} - 6\omega^2 M = cM^2 \tag{4.2}$$

or, equivalently, of the following two real matrix evolution equations:

$$\ddot{U} + 5\omega\dot{V} - 6\omega^2 U = a(U^2 - V^2) - b(UV + VU) \tag{4.3a}$$

$$\ddot{V} - 5\omega\dot{U} - 6\omega^2 V = b(U^2 - V^2) + a(UV + VU) \tag{4.3b}$$

are completely periodic *with period T (isochronous!).*

The singular solutions of (4.2) and (4.3) are not generic, namely—in the context of the initial-value problem—they obtain only for special values of the initial data $M(0)$ and $\dot{M}(0)$, characterized by the requirement to satisfy certain equalities. The difference from the case discussed above (see (2.2) and *lemma 2.1*) is only in the nature of the singularities: simple poles in the previous case, double poles in the one considered here, see (4.2) and (4.3). (Via the 'trick', see (3.3), this is, of course, consistent with the expectation, based on the local analysis of the behaviour of the solution of the nonlinear ODE $W'' = cW^p$, that $W \approx W_0(\tau - \tau_s)^{-\gamma}, \gamma = 2/(p-1)$ in the neighbourhood of a singularity occurring at the value $\tau = \tau_s$).

The proof of this result, which is closely analogous to that of *lemma 2.1*, will be given in a separate paper, together with an analysis of its implications in terms of nonlinear oscillators.

It is easily seen [4] via (an appropriate modification of) the 'trick' (3.3) that the following system of complex evolution equations:

$$\ddot{z}_n - \mathrm{i}\left[\frac{(3+p)}{2}\right]\omega\dot{z}_n - \left[\frac{(1+p)}{2}\right]\omega^2 z_n = \sum_{m_1\ldots m_p=1}^{N} c_{nm_1\ldots m_p}\prod_{l=1}^{p} z_{m_l} \qquad n = 1,\ldots,N \tag{4.4}$$

where N and p are arbitrary positive integers and the N^{p+1} complex constants $c_{nm_1\ldots m_p}$ are arbitrary as well, feature a lot of completely periodic solutions $\underline{z}(t) \equiv (z_1(t),\ldots z_n(t))$ with period T (isochronous: $\underline{z}(t+T) = \underline{z}(t)$), including all those which emerge from an open domain D of initial data $\underline{z}(0), \underline{\dot{z}}(0)$ (in the neighbourhood of the trivial equilibrium solution $\underline{z} = \underline{\dot{z}} = 0$) having positive (nonvanishing!) measure in the $(2N)$-dimensional phase space of such data. The boundary of this domain D is characterized by initial data that yield singular solutions of (4.4). The discussion of the behaviour of the nonsingular solutions of (4.4) that emerge from initial data that fall outside the domain D is an interesting open problem; hints at the rich phenomenology they are likely to feature (including periodic motions with periods that are an integer multiple of T as well as nonperiodic, possibly chaotic, motions) may be evinced from some many-body models that have been recently investigated both analytically and numerically [13, 14].

Note that the result we just stated applies, more generally, to a system characterized by equations of motion of type (4.4) in which the right-hand sides are replaced by arbitrary analytic functions $F_n(\underline{z})$,

$$\ddot{z}_n - \mathrm{i}\left[\frac{(3+p)}{2}\right]\omega\dot{z}_n - \left[\frac{(1+p)}{2}\right]\omega^2 z_n = F_n(\underline{z}) \qquad n = 1,\ldots,N \tag{4.5a}$$

provided these functions $F_n(\underline{z})$ satisfy the scaling relation

$$F_n(\lambda\underline{z}) = \lambda^p F_n(\underline{z}) \tag{4.5b}$$

(the right-hand sides of (4.4) provide an instance of such functions). And, of course, real evolution equations can be obtained from (4.4) or (4.5) in the standard manner, for instance for $p = 3$ (so that the nonlinear part of (4.4) is again cubic) an instance of such equations is the following system of $2N$ nonlinear oscillators characterized by the real Newtonian equations of motion:

$$\begin{aligned}\ddot{\vec{u}}_n + 3\omega\dot{\vec{v}}_n - 2\omega^2\vec{u}_n \\ = \sum_{m_1,m_2,m_3=1}^{N} &\left\{\left(a_{nm_1m_2m_3}\vec{u}_{m_1} - b_{nm_1m_2m_3}\vec{v}_{m_1}\right)\left(\vec{u}_{m_2}\cdot\vec{u}_{m_3} - \vec{v}_{m_2}\cdot\vec{v}_{m_3}\right)\right. \\ &\left. - \left(a_{nm_1m_2m_3}\vec{v}_{m_1} + b_{nm_1m_2m_3}\vec{u}_{m_1}\right)\left(\vec{u}_{m_2}\cdot\vec{v}_{m_3} + \vec{v}_{m_2}\cdot\vec{u}_{m_3}\right)\right\}\end{aligned} \tag{4.6a}$$

$$\ddot{\vec{v}}_n - 3\omega\dot{\vec{u}}_n - 2\omega^2\vec{v}_n$$
$$= \sum_{m_1,m_2,m_3=1}^{N} \left\{ \left(a_{nm_1m_2m_3}\vec{u}_{m_1} - b_{nm_1m_2m_3}\vec{v}_{m_1} \right) \left(\vec{u}_{m_2} \cdot \vec{v}_{m_3} + \vec{v}_{m_2} \cdot \vec{u}_{m_3} \right) \right.$$
$$\left. + \left(a_{nm_1m_2m_3}\vec{v}_{m_1} + b_{nm_1m_2m_3}\vec{u}_{m_1} \right) \left(\vec{u}_{m_2} \cdot \vec{u}_{m_3} - \vec{v}_{m_2} \cdot \vec{v}_{m_3} \right) \right\}. \tag{4.6b}$$

Here N is an arbitrary positive integer, the superimposed arrows denote S-vectors with S also an arbitrary positive integer, dots sandwiched among vectors denote the standard scalar product and the $2N^4$ real constants $a_{nm_1m_2m_3}$, $b_{nm_1m_2m_3}$ are all arbitrary. There is then (see *exercise 5.6.5–19* of [4]) an open domain of initial data $\vec{u}_n(0)$, $\vec{v}_n(0)$, $\dot{\vec{u}}_n(0)$, $\dot{\vec{v}}_n(0)$ in the neighbourhood of the origin in phase space, $\vec{u}_n = \vec{v}_n = \dot{\vec{u}}_n = \dot{\vec{v}}_n = 0$, having nonvanishing measure in the phase space of such data, such that all the motions emerging from it are *completely periodic* with period T, see (2.1):

$$\vec{u}_n(t+T) = \vec{u}_n(t) \qquad \vec{v}_n(t+T) = \vec{v}_n(t). \tag{4.7}$$

But, in contrast to the case of the evolution equations (2.8) and (2.11)—which are clearly special cases of (4.6)—there is no guarantee in this case that *all* the nonsingular solutions of (4.6) be isochronous, namely, that they *all* satisfy (4.7). For instance the system

$$\ddot{z}_1 - 3\mathrm{i}\omega\dot{z}_1 - 2\omega^2 z_1 = 2z_1z_2^2 \tag{4.8a}$$

$$\ddot{z}_2 - 3\mathrm{i}\omega\dot{z}_2 - 2\omega^2 z_2 = 2z_1^2z_2 \tag{4.8b}$$

$$\ddot{z}_3 - 3\mathrm{i}\omega\dot{z}_3 - 2\omega^2 z_3 = (z_1 + z_2)f(\underline{z}) \tag{4.8c}$$

$$\ddot{z}_4 - 3\mathrm{i}\omega\dot{z}_4 - 2\omega^2 z_4 = z_1z_2z_3 \tag{4.8d}$$

where $f(\underline{z})$ is an arbitrary homogeneous polynomial of degree 2 in z_1, z_2, z_3, clearly belongs to the class (4.4) with $p = 3$ (hence to the class (4.6) as well, with $S = 1$, $N = 3$). Yet, as can be easily verified, it possesses the solution

$$z_1(t) = -z_2(t) = \mathrm{i}\omega[1 - a\exp(-\mathrm{i}\omega t)]^{-1} \tag{4.9a}$$

$$z_3(t) = b\exp(\mathrm{i}\omega t) \tag{4.9b}$$

$$z_4(t) = b\exp(\mathrm{i}\omega t)\log\left\{\frac{[\exp(\mathrm{i}\omega t) - a]}{(\mathrm{i}\omega)}\right\} \tag{4.9c}$$

$$z_4(t) = \mathrm{i}b\omega t\exp(\mathrm{i}\omega t)\log\left\{\frac{[1 - a\exp(-\mathrm{i}\omega t)]}{(\mathrm{i}\omega)}\right\} \tag{4.9d}$$

with a, b arbitrary constants. And it is plain that for $b \neq 0$ this solution $\underline{z}(t)$ is, as a function of real t, nonsingular and completely periodic with period T, see (2.1), if $|a| > 1$ (see (4.9*a*), (4.9*b*), (4.9*c*)), singular if $|a| = 1$ (see (4.9*a*), (4.9*c*)) and neither singular nor periodic if $|a| < 1$ (see (4.9*d*) which is, of course, equivalent to (4.9*c*)).

As indicated by example (4.6), there are subcases of the Newtonian equations of motion (4.4) that can be formulated as covariant (hence rotation-invariant) vector equations in a space with an arbitrary number of dimensions; moreover, the property of translation invariance (namely invariance under the transformation $\underline{z}(t) \to \underline{z}(t) + \underline{z}^{(0)}$, with $\underline{z}^{(0)}$ arbitrary but constant, $\dot{\underline{z}}^{(0)} = 0$) can as well be enforced by an appropriate choice of the constants $c_{nm_1\ldots m_p}$ that appear in the right-hand side of (4.4). Let us end this paper by noting that, via an appropriate 'trick' [19, 4], one can also obtain from (2.2) (as well as from all the relevant equations following it in section 2), modified versions of these evolution equations which are translation-invariant and preserve the property that *all* their nonsingular solutions are *completely periodic* with period T (isochronous: see (2.3)). For instance such a modified version of (2.2) reads

$$\ddot{M}^{(\pm)} - \mathrm{i}\left[\frac{(q \pm 3)}{2}\right]\omega\dot{M}^{(+)} - \mathrm{i}\left[\frac{(q \mp 3)}{2}\right]\omega\dot{M}^{(-)} \mp \omega^2(M^{(+)} - M^{(-)}) = \pm\left(\frac{c}{2}\right)(M^{(+)} - M^{(-)})^3 \qquad (4.20)$$

where q is an arbitrary (nonvanishing) integer. Indeed, these two coupled matrix evolution equations for the two dependent variables $M^{(\pm)} \equiv M^{(\pm)}(t)$ (which are clearly invariant under the 'matrix translation' $M^{(\pm)}(t) \to M^{(\pm)}(t) + M^{(0)}$ with $M^{(0)}$ an arbitrary constant matrix, $\dot{M}^{(0)} = 0$) obtain, via the definitions

$$M^{(\pm)} = \frac{(P \pm M)}{2} \qquad P = M^{(+)} + M^{(-)} \qquad M = M^{(+)} - M^{(-)} \qquad (4.21)$$

by supplementing the matrix evolution equation (2.2) with the trivial evolution equation

$$\ddot{P} - \mathrm{i}q\omega\dot{P} = 0 \qquad (4.22a)$$

all solutions of which

$$P(t) = P_0 + P_1\exp(\mathrm{i}q\omega t) \qquad (4.22b)$$

are clearly *completely periodic* with period T, see (2.1).

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